COMBINATORICA
Akadémiai Kiadó – Springer-Verlag

COMBINATORICA **12** (2) (1992) 193–201

# RAMSEY PARTITIONS OF INTEGERS AND FAIR DIVISIONS

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*Received March 10, 1989*

If $k_1$ and $k_2$ are positive integers, the partition $\mathcal{P} = (\alpha_1, \alpha_2, \dots, \alpha_n)$ of $k_1 + k_2$ is said to be a Ramsey partition for the pair $k_1$, $k_2$ if for any sublist $\mathcal{S}$ of $\mathcal{P}$, either there is a sublist of $\mathcal{S}$ which sums to $k_1$ or a sublist of $\mathcal{P} - \mathcal{S}$ which sums to $k_2$. Properties of Ramsey partitions are discussed. In particular it is shown that there is a unique Ramsey partition for $k_1$, $k_2$ having the smallest number $n$ of terms, and in this case $n$ is one more than the sum of the quotients in the Euclidean algorithm for $k_1$ and $k_2$.

An application of Ramsey partitions to the following fair division problem is also discussed: Suppose two persons are to divide a cake fairly in the ratio $k_1 : k_2$. This can be done trivially using $k_1 + k_2 - 1$ cuts. However, every Ramsey partition of $k_1 + k_2$ also yields a fair division algorithm. This method yields fewer cuts except when $k_1 = 1$ and $k_2 = 1$, 2 or 4.

## 1. Introduction

The problem of fairly dividing a cake among $n$ persons apparently originated with Hugo Steinhaus in 1948 [3]. He wrote at the time; "Having found during the war a solution for three partners, I proposed the problem for $n$ partners to B. Knaster and S. Banach." They solved the problem and Steinhaus went on to say; "Interesting mathematical problems arise if we are to determine the minimal number of cuts necessary for fair division." Algorithms have been given requiring $0(n!)$, $0(n^2)$ and $0(n \log n)$ cuts for equal shares [1, 2, 3, 4]. The Moving Knife algorithm requires only $n - 1$ cuts but requires a continuum of simultaneous evaluations by the people doing the dividing.

Suppose now $A$ and $B$ are to share a cake with their portions in the ratio of $k_1$ to $k_2$. Thus we seek a partition of the cake $X = X_1 \cup X_2$ with $\mu_A(X_1) \geq k_1/(k_1 + k_2)$ and $\mu_B(X_2) \geq k_2/(k_1 + k_2)$ where $\mu_A$ and $\mu_B$ are the probability measures on $X$ by which $A$ and $B$ evaluate pieces of the cake. A solution is given by assuming there are $k_1 + k_2$ players who are to get equal shares with the additional information that their individual probability measures by which the pieces are evaluated satisfy $\mu_1 = \mu_2 = \ldots = \mu_{k_1} = \mu_A$ and $\mu_{k_1+1} = \ldots = \mu_{k_1+k_2} = \mu_B$. The least number of cuts required by known algorithms would be $k_1 + k_2 - 1$ which is realized by the Moving Knife solution. Letting $A$ cut $k_1 + k_2$ equal pieces and $B$ choose $k_2$ of them would also accomplish the task in $k_1 + k_2 - 1$ cuts.

In what follows we identify special partitions of the integer $k_1 + k_2$, which are used for fair division in the ratio $k_1 : k_2$. We first study these partitions for the pair $k_1$, $k_2$, which are interesting in their own right, before showing how they are used to accomplish fair divisions with fewer than $k_1 + k_2 - 1$ cuts for two persons who are to receive unequal shares.

AMS subject classification code (1991): 05 A 17, 05 D 10, 11 P 81

## 2. Ramsey Partitions of Integers

Assume throughout that $\mathcal{P} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ where all $\alpha_i$ are positive integers and $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$. In this case $\mathcal{P}$ partitions the integer $\sum_{i=1}^{n} \alpha_i$. If $\mathcal{S} = (\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_r})$, $i_1 < i_2 < \ldots < i_r$ is a sublist of $\mathcal{P}$ we will write $\mathcal{S} \subset \mathcal{P}$. The complementary sublist is denoted by $\mathcal{P} - \mathcal{S}$.

**Definition 2.1.** A partition $\mathcal{P} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is *a Ramsey partition for the pair* $k_1$, $k_2$ provided $\sum_{i=1}^{n} \alpha_i = k_1 + k_2$ and for any $\mathcal{S} \subset \mathcal{P}$ either

(*i*) $$\sum_{\mathcal{S}'} \alpha_i = k_1 \text{ for some } \mathcal{S}' \subset \mathcal{S} \text{ or}$$

(*ii*) $$\sum_{\mathcal{S}'} \alpha_i = k_2 \text{ for some } \mathcal{S}' \subset \mathcal{P} - \mathcal{S}.$$

See Table 1 for examples.

**Proposition 2.2.** *The following three conditions are equivalent.*
(1) *$\mathcal{P}$ is Ramsey for the pair $k_1$, $k_2$.*
(2) *For any $\mathcal{S} \subset \mathcal{P}$, $\sum_{\mathcal{S}} \alpha_i \geq k_l$ implies for some $\mathcal{S}' \subset \mathcal{S}$, $\sum_{\mathcal{S}'} \alpha_i = k_l$, for $l = 1, 2$.*
(3) *Any $i_1 < i_2 < \ldots < i_t$ satisfying $\alpha_{i_1} + \alpha_{i_2} + \ldots + \alpha_{i_{t-1}} < k_l$ and $\alpha_{i_1} + \alpha_{i_2} + \ldots + \alpha_{i_t} \geq k_l$ requires $\alpha_{i_1} + \alpha_{i_2} + \ldots + \alpha_{i_t} = k_l$, for $l = 1, 2$.*

**Proof.** We show (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). If $\sum_{\mathcal{S}} \alpha_i > k_1$ and $\sum_{\mathcal{S}'} \alpha_i \neq k_1$ for all $\mathcal{S}' \subset \mathcal{S}$ then neither (*i*) nor (*ii*) can hold for $\mathcal{S}$ since $\sum_{\mathcal{P}-\mathcal{S}} \alpha_i < k_2$. If $\alpha_{i_1} + \alpha_{i_2} + \ldots + \alpha_{i_{t-1}} < k_1$ while $\alpha_{i_1} + \alpha_{i_2} + \ldots + \alpha_{i_t} > k_1$, then (2) fails for $l = 1$ and $\mathcal{S} = (\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_t})$ since the $\alpha_{i_j}$ are non-increasing. The Ramsey condition follows immediately from (3) since for any $S$ we have $\sum_{\mathcal{S}} \alpha_i \geq k_1$, or $\sum_{\mathcal{P}-\mathcal{S}} \alpha_i \geq k_2$. ∎

**Corollary 2.3.** *If $\mathcal{P}$ is Ramsey for the pair $k_1$, $k_2$ with $k_1 \leq k_2$ then*
(1) *No term of $\mathcal{P}$ exceeds $k_1$.*
(2) *$\mathcal{P}$ does not contain exactly $k_1 - 1$ or $k_2 - 1$ ones.*
(3) *For some $j_1$ and $j_2$, $\alpha_1, \ldots, \alpha_{j_1}$ partition $k_1$, $\alpha_{j_1+1}, \ldots, \alpha_n$ partition $k_2$, $\alpha_1, \ldots, \alpha_{j_2}$ partition $k_2$ and $\alpha_{j_2+1}, \ldots, \alpha_n$ partition $k_1$.*
(4) *With $j_1$ as in (3), $\mathcal{P}' = (k_1, \alpha_{j_1+1}, \ldots, \alpha_n)$ is also Ramsey for the pair $k_1$, $k_2$.*

**Comment.** None of the conditions of 2.3 is sufficient for a Ramsey partition; $\mathcal{P} = (5, 3, 2, 2, 2, 2)$ satisfies (1), (2), (3), and the corresponding $\mathcal{P}'$ in (4) is Ramsey for the pair 8, 10, yet $\mathcal{P}$ is not Ramsey for 8, 10.

**Definition 2.4.** Partition $\mathcal{P} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a *minimal Ramsey partition for the pair* $k_1$, $k_2$ if whenever $\mathcal{P}' = (\beta_1, \beta_2, \ldots, \beta_m)$ is also Ramsey for the pair then $n \leq m$.

We will show that the minimal Ramsey partition for the pair $k_1$, $k_2$ is unique. It must have $\alpha_1 = \min\{k_1, k_2\}$ from 2.3 (4). The following will provide a method of constructing this special partition.

1. ( 8, 5, 3, 2, 1, 1, 1)
2. ( 8, 5, 3, 1, 1, 1, 1, 1)
3. ( 8, 5, 2, 1, 1, 1, 1, 1, 1)
4. ( 8, 5, 1, 1, 1, 1, 1, 1, 1, 1)
5. ( 8, 4, 1, 1, 1, 1, 1, 1, 1, 1, 1)
6. ( 8, 3, 2, 2, 1, 1, 1, 1, 1, 1)
7. ( 8, 3, 2, 1, 1, 1, 1, 1, 1, 1, 1)
8. ( 8, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
9. ( 8, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1)
10. ( 8, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
11. ( 8, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
12. ( 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
13. ( 6, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1)
14. ( 6, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
15. ( 6, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
16. ( 6, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
17. ( 5, 3, 3, 2, 1, 1, 1, 1, 1, 1, 1, 1)
18. ( 5, 3, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
19. ( 5, 3, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
20. ( 5, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
21. ( 5, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
22. ( 5, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
23. ( 4, 4, 4, 1, 1, 1, 1, 1, 1, 1, 1, 1)
24. ( 4, 4, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
25. ( 4, 4, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1)
26. ( 4, 4, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
27. ( 4, 4, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
28. ( 4, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
29. ( 4, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1)
30. ( 4, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
31. ( 4, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
32. ( 4, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
33. ( 4, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
34. ( 3, 3, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
35. ( 3, 3, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
36. ( 3, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
37. ( 3, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
38. ( 3, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
39. ( 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
40. ( 2, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1)
41. ( 2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
42. ( 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
43. ( 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
44. ( 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
45. ( 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
46. ( 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)

*Table 1.* The 46 Ramsey Partitions for the Pair 8, 13

**Theorem 2.5.** *Let $\mathcal{P} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ partition $k_2$ so that $\mathcal{P}' = (k_1, \alpha_1, \ldots, \alpha_n)$ partitions $k_1 + k_2$ with $\alpha_1 \leq k_1 \leq k_2$. Then*

(1) *$\mathcal{P}'$ is Ramsey for the pair $k_1$, $k_2$ if and only if $\mathcal{P}$ is Ramsey for the pair $k_1$, $k_2 - k_1$ and*

(2) *$\mathcal{P}'$ is minimal Ramsey for the pair $k_1$, $k_2$ if and only if $\mathcal{P}$ is minimal Ramsey for the pair $k_1$, $k_2 - k_1$. (No assumption is made on the relative size of $k_2 - k_1$ and $k_1$).*

**Proof.** Sums from $\mathcal{P}$ as in 2.2 (3) would skip $k_1$ if and only if the same were true for $\mathcal{P}'$.
Also if terms from $\mathcal{P}$ satisfy $\alpha_{i_1}+\alpha_{i_2}+\ldots+\alpha_{i_{t-1}} < k_2-k_1$ and $\alpha_{i_1}+\alpha_{i_2}+\ldots+\alpha_{i_t} > k_2-k_1$ then terms from $\mathcal{P}'$ satisfy $k_1+\alpha_{i_1}+\ldots+\alpha_{i_{t-1}} < k_2$ and $k_1+\alpha_{i_1}+\ldots+\alpha_{i_t} > k_2$. Conversely for a sum in $\mathcal{P}'$ to skip $k_2$, since $\alpha_1+\alpha_2+\ldots+\alpha_n = k_2$, we must have $k_1+\alpha_{i_1}+\alpha_{i_2}+\ldots+\alpha_{i_{t-1}} < k_2$ and $k_1+\alpha_{i_1}+\alpha_{i_2}+\ldots+\alpha_{i_t} > k_2$ which requires $\alpha_{i_1}+\alpha_{i_2}+\ldots+\alpha_{i_{t-1}} < k_2-k_1$, and $\alpha_{i_1}+\alpha_{i_2}+\ldots+\alpha_{i_t} > k_2-k_1$ in $\mathcal{P}$.

As noted above, if $\mathcal{P}'$ is minimal Ramsey for $k_1$, $k_2$ then its first term must be $k_1$, and $(k_1,\beta_1,\beta_2,\ldots,\beta_r), r<n$, is Ramsey for $k_1$, $k_2$ if and only if $(\beta_1,\beta_2,\ldots,\beta_r)$ is Ramsey for $k_1$, $k_2-k_1$. Thus the partitions are either both minimal or both non-minimal. ∎

We can now use Theorem 2.5 to construct the unique minimal Ramsey partition $\mathcal{P}$ for the pair $k_1$, $k_2-k_1$. Assuming $k_1 \le k_2$, the first entry of $\mathcal{P}$ must be $k_1$, and what remains must be minimal Ramsey for $k_2-k_1$, $k_1$. Thus the second entry is $\min\{k_1, k_2-k_1\}$ and the process is repeated. If $\gcd(k_1,k_2)=d$ the process reaches the final stage of requiring a minimal Ramsey partition for $d$, $md$ for some integer $m$, which is readily seen to be $m+1$ entries of $d$.

**Proposition 2.6.** *There is a unique minimal Ramsey partition $\mathcal{P}=(\alpha_1,\alpha_2,\ldots,\alpha_n)$ for the pair $k_1$, $k_2$. If $\mathcal{P}'=(\beta_1,\beta_2,\ldots,\beta_m)$ is any Ramsey partition for $k_1$, $k_2$ then for some $s_1<s_2<\ldots<s_{n-1}$ we have:*

$$\begin{cases} \beta_1+\beta_2+\ldots+\beta_{s_1}=\alpha_1 \\ \beta_{s_1+1}+\ldots+\beta_{s_2}=\alpha_2 \\ \vdots \\ \beta_{s_{n-1}+1}+\ldots+\beta_m=\alpha_n \end{cases}$$

**Proof.** We have seen above there is an algorithm which produces the unique minimal partition. We know also $\alpha_1=k_1$ (assuming $k_1\le k_2$) and from 2.2 (3) $\beta_1+\beta_2+\ldots+\beta_{s_1}=\alpha_1$ for some $s_1$. If $n=2$ we are done. Otherwise from Theorem 2.5, $\mathcal{P}_1=(\alpha_2,\alpha_3,\ldots,\alpha_n)$ is minimal Ramsey for $k_2-k_1$, $k_1$ while $\mathcal{P}_1'=(\beta_{s_1+1},\beta_{s_1+2},\ldots,\beta_m)$ is Ramsey for $k_2-k_1$, $k_1$ and the argument is repeated. ∎

Thus the unique minimal Ramsey partition for the pair $k_1$, $k_2$ has strictly fewer entries than any other Ramsey partition for the pair. We set $f(k_1,k_2)$ equal to one less than the number of entries in the minimal Ramsey partition for the pair $k_1$, $k_2$. This is the number of "cuts" one must make to produce the partition.

**Theorem 2.7.** *The number $f(k_1,k_2)$ is the sum of the quotients in the Euclidean algorithm for $k_1$ and $k_2$. If $(\beta_1,\beta_2,\ldots,\beta_m)$ is any other Ramsey partition for $k_1$, $k_2$ we have $m>f(k_1,k_2)+1$.*

**Proof.** For example $f(8,22)=6=2+1+3$ since $22=2\cdot 8+6, 8=1\cdot 6+2$ and $6=3\cdot 2+0$. The corresponding minimal Ramsey partition for 8, 22 is (8,8,6,2,2,2,2). The second statement has been observed above so we look at the first. If $f(k_1,k_2)=1$ then $k_1=k_2=d$ for some $d$ and the minimal Ramsey partition is $(d,d)$ and one is the sum of the quotients for $k_1$ and $k_2$. Assume the result for any pair with $f(k_1,k_2)\le j-1$ and suppose a new pair $k_1$, $k_2$ with $k_1\le k_2$ has $j+1$ entries in the corresponding minimal Ramsey partition.

**Case 1:** $k_2 \geq 2k_1$

From 2.5 we have $f(k_1, k_2) = 1 + f(k_1, k_2 - k_1)$. But the first quotient for $k_1$, $k_2 - k_1$ is one less than that for $k_1$, $k_2$ while all other quotients are the same. The induction hypothesis applied to the pair $k_1$, $k_2 - k_1$ gives the required result for the pair $k_1$, $k_2$.

**Case 2:** $k_2 < 2k_1$

As before, $f(k_1, k_2) = 1 + f(k_1, k_2 - k_1) = 1 + f(k_2 - k_1, k_1)$. The first quotient for $k_1$ and $k_2$ is one and the first remainder is $k_2 - k_1$. Thus the rest of the Euclidean algorithm for $k_1$, $k_2$ is the same as the algorithm for $k_2 - k_1$, $k_1$. The result follows. ∎

**Proposition 2.8.** *Suppose $k_1 < k_2$, then*

(1) $f(k_1, k_2) = f(k_2 - k_1, k_2)$,

(2) $f(1, k_2) = k_2$,

(3) $f(mk_1, mk_2) = f(k_1, k_2)$

(4) $f(k_1, k_2) \leq \left\lceil \dfrac{k_1 + k_2}{2} \right\rceil$ *if* $k_1 \geq 2$,

(5) $\left\lceil \dfrac{k_2}{k_1} \right\rceil \leq f(k_1, k_2)$ *for all* $k_1$, $k_2$, *and*

(6) $\lim\limits_{k_2 \to \infty} \dfrac{f(k_1, k_2)}{(\frac{k_2}{k_1})} = 1$ *for all* $k_1$.

**Proof.** We know $f(k_1, k_2) = 1 + f(k_1, k_2 - k_1) = f(k_2 - k_1, k_2)$, both equalities from 2.5 since $(k_1, k_2 - k_1) = (k_2 - (k_2 - k_1), k_2 - k_1)$. The minimal Ramsey partition for $1, k_2$ has $k_2 + 1$ ones. For (3), the Euclidean algorithm for $k_1$ and $k_2$ has the same quotients as for $mk_1$, $mk_2$.

To see (4), if $k_1 \geq 2$ and $k_1 = k_2$, the minimal Ramsey partition is $(k_1, k_1)$ so $f(k_1, k_2) = 1 < (k_1 + k_2)/2$. Also, if $k_2 = k_1 + 1$, the minimal Ramsey partition is $(k_1, 1, 1, \ldots, 1)$ so $f(k_1, k_2) = k_1 + 1 = \left\lceil \dfrac{2k_1 + 1}{2} \right\rceil = \left\lceil \dfrac{k_1 + k_2}{2} \right\rceil$.

Now assume (4) whenever $k_1 + k_2 \leq m$, $2 \leq k_1 \leq k_2$. Then where $k_1 + k_2 = m + 1$, $2 \leq k_1 \leq k_2$, and $k_2 - k_1 \geq 2$ we observe $f(k_1, k_2) = 1 + f(k_2 - k_1, k_1) \leq 1 + \left\lceil \frac{k_2}{2} \right\rceil = \left\lceil \frac{2 + k_2}{2} \right\rceil \leq \left\lceil \frac{k_1 + k_2}{2} \right\rceil$, the first inequality following from the induction hypothesis.

The equality (5) is immediate since we know that each term of the partition of $k_1 + k_2$ is no larger than $k_1$. In (6) the numerator is $\lfloor \frac{k_2}{k_1} \rfloor$ plus $f(j, k_1)$ where $j = k_2 \bmod k_1$. For all $j$ the numerator is between $(k_2/k_1)$ and $(k_2/k_1) + 2k_1$, since the longest Ramsey partition for the pair $j$, $k_1$ is all ones. ∎

It appears to be difficult to list all Ramsey partitions for a pair $k_1$, $k_2$ or to determine how many there are. We will address this number-theoretic question in later work. Table 1 gives the 46 Ramsey partitions for the pair 8, 13 while Table 2 gives the <u>number</u> of Ramsey partitions for various pairs $k_1$, $k_2$.

| $k_2 \backslash k_1$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 3 | | | | | | | | | | | |
| 3 | 2 | 5 | | | | | | | | | | |
| 4 | 4 | 3 | 11 | | | | | | | | | |
| 5 | 3 | 4 | 5 | 14 | | | | | | | | |
| 6 | 5 | 7 | 11 | 7 | 31 | | | | | | | |
| 7 | 4 | 5 | 8 | 10 | 11 | 34 | | | | | | |
| 8 | 6 | 6 | 19 | 11 | 24 | 15 | 69 | | | | | |
| 9 | 5 | 9 | 11 | 12 | 25 | 22 | 22 | 82 | | | | |
| 10 | 7 | 7 | 20 | 21 | 33 | 23 | 49 | 30 | 139 | | | |
| 11 | 6 | 8 | 15 | 14 | 24 | 27 | 36 | 45 | 42 | 149 | | |
| 12 | 8 | 11 | 29 | 17 | 61 | 29 | 79 | 60 | 90 | 56 | 302 | |
| 13 | 7 | 9 | 19 | 18 | 31 | 30 | 46 | 55 | 66 | 84 | 77 | 280 |
| 14 | 9 | 10 | 31 | 19 | 53 | 53 | 84 | 57 | 120 | 86 | 166 | 101 |
| 15 | 8 | 13 | 24 | 28 | 53 | 34 | 56 | 85 | 103 | 102 | 148 | |
| 16 | 10 | 11 | 41 | 21 | 65 | 41 | 146 | 70 | 144 | 99 | | |
| 17 | 9 | 12 | 29 | 24 | 51 | 42 | 69 | 72 | 102 | | | |
| 18 | 11 | 15 | 44 | 25 | 101 | 46 | 126 | 146 | | | | |
| 19 | 10 | 13 | 35 | 26 | 61 | 48 | 90 | | | | | |
| 20 | 12 | 14 | 55 | 35 | 92 | 49 | | | | | | |
| 21 | 11 | 17 | 41 | 28 | 91 | | | | | | | |
| 22 | 13 | 15 | 59 | 31 | | | | | | | | |
| 23 | 12 | 16 | 48 | | | | | | | | | |
| 24 | 14 | 19 | | | | | | | | | | |
| 25 | 13 | | | | | | | | | | | |

*Table 2.* The Number of Ramsey Partitions for Various $k_1$, $k_2$

## 3. Fair Divisions Using Ramsey Partitions

We now use Ramsey partitions to construct algorithms for fair division when two persons are to get shares in the ratio of $k_1$ to $k_2$. Since the problem for $k_1 : k_2$ is the same as for $ak_1 : ak_2$ we will assume henceforth that $k_1$ and $k_2$ and relatively prime, whence $k_1 < k_2$ unless $k_1 = k_2 = 1$. These algorithms require fewer cuts than $k_1 + k_2 - 1$ except when $k_1 = 1$ and $k_2 = 1, 2$ or 4.

### 3.1. Fair Division Algorithm by Ramsey Partitions for Two Persons Receiving Unequal Shares

Given persons $A$ and $B$, probability measures $\mu_A$ and $\mu_B$, and integers $1 \le k_1 \le k_2$, to partition a cake $X = S_1 \cup S_2$ so that $\mu_A(S_1) \ge k_1/(k_1 + k_2)$ and $\mu_B(S_2) \ge k_2/(k_1 + k_2)$:

1. Let $\mathcal{P} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be a Ramsey partition for the pair $k_1$, $k_2$.
2. Let $A$ cut $X = X_1 \cup X_2 \cup \ldots \cup X_m$ so that $\mu_A(X_i) = \alpha_i/(k_1 + k_2)$.

3. Set $\mathcal{S} = (\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_r})$ where $\alpha_{i_j}$ appears in $\mathcal{S}$ if and only if $\mu_B(X_{i_j}) \geq \alpha_{i_j}/(k_1 + k_2)$.
4. If some sublist of $\mathcal{S}$ sums to $k_2$ assign to $B$ the corresponding pieces; otherwise assign to $A$ pieces corresponding to a sublist in $\mathcal{P} - \mathcal{S}$ which sums to $k_1$. In each case the second person receives the remaining pieces.

It is clear the algorithm generates the necessary sets $S_1$ and $S_2$ using $f(k_1, k_2)$ cuts when the minimal Ramsey partition is used. We wish this to be fewer than $k_1 + k_2 - 1$ to improve on known algorithms. By 2.8 (4) when $k_2 > k_1 \geq 2$ then at most $\left\lceil \frac{k_1+k_2}{2} \right\rceil$ cuts are required and with $k_1$ and $k_2$ relatively prime this improves on $k_1 + k_2 - 1$. If $k_1 = 1$ the only Ramsey partition consists of $k_1 + k_2$ ones which makes $f(k_1, k_2) = k_1 + k_2 - 1$. To improve on the number of cuts in this and other cases we reduce the original problem to a smaller one.

**3.2. Reduced Fair Division Algorithm.**

The problem is the same as that for 3.1. We assume $k_1 < k_2$.

1. Let $A$ cut $X = X_1 \cup X_2$ where $\mu_A(X_1) = \mu_A(X_2) = \frac{1}{2}$.
2. Let $B$ choose, say $X_2$, with $\mu_B(X_2) \geq \frac{1}{2}$.
3. Implement Algorithm 3.1 on $X_1$ for $A$ and $B$ in the new ratio $k_1$, $(k_2 - k_1)/2$ (or $2k_1$, $k_2 - k_1$).

**Example 3.3 Divide the Cake in the Ratio 1:3.**

With $k_1 = 1$, $k_2 = 3$, $A$ cuts halves $X_1$, $X_2$ and assume $B$ chooses $X_2$. Then $X_1$ is divided in the ratio of 1 to $\frac{3-1}{2} = 1$ with one additional cut. Two cuts were used instead of $f(1, 3) = 3$ cuts as required in 3.1.

The following theorem establishes that proper portions are given by 3.2 and that for $k_1 = 1$ the algorithm requires fewer than $f(1, k_2) = k_2$ cuts for ratio $1, k_2$ when $k_2$ is not 1,2, or 4. This and 2.8 (3) and (4) show that division into ratio $k_1 : k_2$ using 3.1 or 3.2 requires fewer cuts than known algorithms (including "Moving Knife" for $k_1 + k_2$ persons) unless $k_1 = 1$ and $k_2 = 1$ (one cut), $k_2 = 2$ (two cuts) or $k_2 = 4$ (four cuts).

**Theorem 3.4.** *The reduced Fair Division Algorithm accomplishes fair division for the unequal portions in the ratio $k_1 : k_2$. If $1 = k_1 < k_2$ and $k_2 \neq 2$ or 4, then fewer than $f(1, k_2) = k_2$ cuts are required.*

**Proof.** The algorithm assigns $A$ pieces composing $S_1$ with $\mu_A(S_1) = [k_1/(\frac{k_1+k_2}{2})] \cdot \frac{1}{2} = k_1/(k_1 + k_2)$ while $B$ gets pieces composing $S_2$ with

$$\mu_B(S_2) \geq \frac{1}{2} + [(\frac{k_2 - k_1}{2})/(\frac{k_1 + k_2}{2})] \cdot \frac{1}{2} = k_2/(k_1 + k_2)$$

as required. The number of cuts used is $1 + f(2k_1, k_2 - k_1)$. Setting $k_1 = 1$, when $k_2 - k_1$ is even we have $1 + f(1, \frac{k_2-1}{2}) = 1 + (k_2 - 1)/2 = (1 + k_2)/2$ cuts which is less than $f(1, k_2) = k_2$ since $k_2 > 1$. When $k_2 - k_1$ is odd and $k_1 = 1$ the number of cuts is $1 + f(2k_1, k_2 - k_1) = 1 + f(2, k_2 - 1) \leq 1 + \left\lceil \frac{k_2+1}{2} \right\rceil = \left\lceil \frac{k_2+3}{2} \right\rceil$ which is less than $f(1, k_2) = k_2$ whenever $k_2 > 4$. ∎

Comparing numbers $f(k_1, k_2)$ and $1 + f(2k_1, k_2 - k_1)$ shows when 3.2 improves on the number of cuts required by 3.1. In some cases reductions are substantial. Also one can reduce by other than halves. The following example shows how reduction by thirds can reduce the number of cuts.

**Example 3.5. Dividing the Cake in the Ratio of 7:8.**

The minimal Ramsey partition for 7, 8 is (7,1,1,1,1,1,1,1,1) requiring 8 cuts using 3.1 Reduction by thirds reduces the number of cuts to 5. Have $A$ cut thirds $X_1 \cup X_2 \cup X_3$ and have $B$ choose one piece for himself, say $X_3$ and assign one piece to $A$, say $X_2$. Then $X_1$ should be divided between $A$ and $B$ in the ratio of 2 to 3 to accomplish the overall 7, 8 division. The minimal Ramsey partition for 2, 3 is (2,1,1,1) requiring 3 cuts.

Finally we examine the situation for unequal shares for more than two persons. The algorithm is inductive.

**3.6. Fair Division Algorithm for $n$ Persons, Unequal Shares**

Given persons $P_1$, $P_2, \dots, P_n$ with corresponding probability measures $\mu_1, \mu_2, \dots, \mu_n$ and positive integers $k_1$, $k_2, \dots, k_n$ with $\sum_{i=1}^{n} k_i = K$, to partition a cake $X = X_1 \cup X_2 \cup \dots \cup X_n$ so that $\mu_i(X_i) \geq k_i/K$ for $i = 1, 2, \dots, n$:

1. Assuming a solution for $n - 1$ players, partition the cake $X = X'_1 \cup X'_2 \cup \dots \cup X'_{n-1}$ so that $\mu_i(X'_i) \geq k_i/(K - k_n)$, $i = 1, 2, \dots, n - 1$.
2. For each $i = 1, 2, \dots, n - 1$ execute 3.1 for persons $P_i$ and $P_n$ in the ratio $K - k_n : k_n$ on $X'_i$, giving $P_n$ a portion from each $X'_i$; the remainder of $X'_i$ comprises $X_i$.

Let $f(k_1, k_2, \dots, k_n)$ denote the number of cuts required in implementing 3.6.

**Theorem 3.8.** *Algorithm 3.6 accomplishes fair division in the ratios $k_1 : k_2 : \dots : k_n$. The number of cuts used follows the recursion $f(k_1, k_2, \dots, k_n) = f(k_1, k_2, \dots, k_{n-1}) + (n - 1)f(k_1 + k_2 + \dots + k_{n-1}, k_n)$.*

**Proof.** Person $P_i$ will get at least $\frac{K-k_i}{K} \cdot \frac{k_i}{K-k_i} = \frac{k_i}{K}$, $i = 1, 2, \dots, n - 1$ while person $P_n$ gets at least $\frac{k_n}{K}(\mu_n(X'_1) + \dots + \mu_n(X'_{n-1})) = \frac{k_n}{K}$ as required. The recursion formula is immediate.

**Example 3.7.** Persons $P_1$, $P_2$, $P_3$ are to receive portions in the ratio $1 : 3 : 10$. First $P_1$ and $P_2$ divide the cake in the ratio of $1 : 3$ ising 3.2 which requires 2 cuts as in Example 3.3. Then both $P_1$ and $P_2$ divide their portions with $P_3$ in the ratio of $4 : 10$ or $2 : 5$ which requires 4 cuts each. Ten cuts total are used, whereas the Moving Knife algorithm requires 13. Person $P_1$ gets a portion at least $\frac{2}{7} \cdot \frac{1}{4} = \frac{1}{14}$, $P_2$ gets at least $\frac{2}{7} \cdot \frac{3}{4} = \frac{3}{14}$ while $P_3$ gets at least $\frac{5}{7}(\frac{1}{4} + \frac{3}{4}) = \frac{5}{7}$ as required. ∎

In the case $k_1 = k_2 = \dots = k_n = 1$ the number of cuts required by 3.6 is $f(k_1, k_2, \dots, k_n) = f(k_1, k_2, \dots, k_{n-1}) + (n - 1)f(k_1 + k_2 + \dots + k_{n-1}, 1) = f(k_1, k_2, \dots, k_{n-1}) + (n - 1)^2 = \dots = 1^2 + 2^2 + 3^2 + \dots + (n - 1)^2$ if 3.1 but not 3.2 is used. In this case the number of cuts required by 3.6 is $0(n^3)$ so that more cuts are required for large $n$ than known algorithms for equal shares requiring $o(n^2)$ or $0(n \log n)$ cuts. However, the use of 3.2 reduces the number of cuts and as 3.7

shows, for unequal shares the number of cuts can be less than $K - 1$ improving on the number of cuts required even for "Moving Knife."

**Comment:** Ramsey partitions could be defined for the sequence $k_1, k_2, \ldots, k_n$ by imposing condition 2.2 (3) for $l = 1, 2, \ldots, n$. Questions concerning such special partitions of $\sum_{i=1}^{n} k_i$ are interesting from the number-theoretic point of view. Can they be used in more general settings of fair divisions, and do they have other applications?

**Acknowledgement.** The authors appreciate the computing necessary for Tables 1 and 2 done by Mr. Robert Hale of Deakin University, and the help of the referee in the preparation of the manuscript.

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